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Minimum Logic of the Whole (MLW)

CONTENTS

1. Introduction

There exist many mereological systems dealing with the notions of "part" and "whole". Nevertheless we need not only a logic about bigger or less parts and their Boolean relations, but a "logic of the whole", where an effect of so called "holistic property", property of the whole, could be adequately expressed. The paper is an effort to build a minimal logical framework for this "logic of the whole". The paper deals with the semiformal primary representation of the phenomenon of "holistic properties". A Minimum Logic of the Whole (MLW) is described as a logical system with two-level order. Axioms, some definitions, theorems, and models of MLW are given below.

2. Basic concepts of MLW

Suppose $A \leq B$ is a relation of nonstrong order; then by definition, put

(E) $A = B \;\equiv\; A \leq B \wedge B \leq A$,
where $A = B$ is an equality between A and B.

Therefore $\leq$ is a relation with the following properties.
1. $A \leq A$, the property of reflexivity,
2. $A \leq B$ and $B \leq A$ entails $A = B$, the property of antisymmetry,
3. $A \leq B$ and $B \leq C$ entails $A \leq C$, the property of transitivity.
For example, we shall have the following definitions:

(Min) $\mathrm{Min}(A) \equiv A \leq A \wedge \forall B(B \leq B \supset A \leq B)$, i.e., A is a minimum element.

(Max) $\mathrm{Max}(A) \equiv A \leq A \wedge \forall B(B \leq B \supset B \leq A)$, i.e., A is a maximum element.

(P) $\mathrm{Pos}(A) \equiv A \leq A \wedge \exists B(B \leq A \wedge \neg(A \leq B))$, i.e., A is a positive element.

(At) $\mathrm{At}(A) \equiv \mathrm{Pos}(A) \wedge \forall B(B \leq A \wedge \mathrm{Pos}(B) \supset A \leq B)$ – A is an atom.

Let $A \leq^1 B$ and $A \leq^2 B$ be two nonstrong orders also. We can define the same concepts for the relations $\leq^1$ and $\leq^2$, only with symbols "1" or "2" in appropriate definitions. For example, we have

(E^1) $A =^1 B \equiv A \leq^1 B \wedge B \leq^1 A$, i.e., A 1-equals B.

(Min^2) $\mathrm{Min}^2(A) \equiv A \leq^2 A \wedge \forall B(B \leq^2 B \supset A \leq^2 B)$, i.e., A is a 2-minimum element.

(P^2) $\mathrm{Pos}^2(A) \equiv A \leq^2 A \wedge \exists B(B \leq^2 A \wedge \neg(A \leq^2 B))$, i.e., A is a 2-positive element.

(At^2) $\mathrm{At}^2(A) \equiv \mathrm{Pos}^2(A) \wedge \forall B(B \leq^2 A \wedge \mathrm{Pos}^2(B) \supset A \leq^2 B)$, i.e., A is a 2-atom.

Consider the following two axioms.

(AH1) $A \leq^i B \supset A \leq B$, where i=1,2, i.e., i-orders entail the order $\leq$.

(AH2) $\forall X(\mathrm{Pos}^2(X) \supset \exists Y(\mathrm{Pos}^1(Y) \wedge Y \leq X)) \wedge \forall X \forall Y(\mathrm{Pos}^2(X) \wedge \mathrm{Pos}^1(Y) \supset \neg(X \leq Y))$, i.e., any 2-positive element contains a 1-positive element and no one 1-positive element contains a 2-positive element.

We shall say that a logical system with only three primary predicates $\leq$, $\leq^1$, and $\leq^2$, presented above, and with only nonlogical axioms (AH1), (AH2), is called a *Minimum Logic of the Whole* (MLW). System of the orders $\leq$, $\leq^1$, and $\leq^2$, presented above, is called *two-level order* and is denoted by $\leq(\leq^1, \leq^2)$.

Therefore MLW is a logic with two-level order $\leq(\leq^1, \leq^2)$, where $\leq^1$ is an order of the first level (1-order) and $\leq^2$ is an order of the second level (2-order). Every of the orders is defined only in the own level. Otherwise the relation $\leq$, as it follows from (AH1), is a "trans-level" order, defining in both levels. The first level is the level of parts and elements. The second level is the level of wholes. Axiom (AH2) determines the relation between these levels. Any whole contains a part and no part contains a whole. More precisely, the following definitions can be given.

(DH) $H(X) \equiv \mathrm{Pos}^2(X)$, where H(X) is "X is a whole".

Therefore a whole is a 2-positive element, i.e., not null element of the second level.

(DP) $PP(Y,X) \equiv Pos^2(X) \wedge Pos^1(Y) \wedge (Y \leq X)$, where $PP(Y,X)$ is "Y is a proper part of a whole X"

The following theorem, Theorem of the Whole (TW), can be proved here.

(TW) $PP(Y,X) \supset \neg(X = Y)$, i.e., if Y is a proper part of whole X, then X does not equal Y.

Proof. $PP(Y,X)$ entails $Pos^2(X) \wedge Pos^1(Y)$. Further, in accordance with (AH2), we obtain $\neg(X \leq Y)$. Therefore we have $\neg(X = Y)$. This completes the proof of (TW).

The property Pos^2 is called a *holistic property* of 2-level. We have that

$H(X) \supset Pos^2(X)$, i.e., wholes have holistic properties, and

$Pos^1(Y) \supset \neg Pos^2(Y)$, i.e., 1-positive elements have not holistic properties.

3. Models of MLW

Let us consider some interpretations of MLW.

1. Suppose X, Y, Z … are sets from a class K, both finite and infinite. Let us introduce the following predicates on sets.

KSet(X) is "X is an element of K"

FinSet(X) is the assertion "X is a finite set and X is an element of K". Suppose null set ∅ belongs to K and ∅ is a finite set too.

InfSet(X) is the assertion "X is an infinite set and X is an element of K".

Put by definition

$X \leq Y \equiv X \subseteq Y \wedge KSet(X) \wedge KSet(Y)$, where «$X \subseteq Y$» denotes "X is a subset of Y".

$X \leq^1 Y \equiv X \subseteq Y \wedge FinSet(X) \wedge FinSet(Y)$.

$X \leq^2 Y \equiv X \subseteq Y \wedge (InfSet(X) \vee X= \varnothing) \wedge (InfSet(Y) \vee Y= \varnothing)$.

Therefore, 1-level is the class of all finite sets from K. 2-level is the class of all infinite sets from K and null set. The reader will easily prove that axioms (AH1) and (AH2) hold here. Besides, we have

Theorem of Inf-whole. $Pos^2(X) \equiv InfSet(X)$

Therefore wholes are infinite sets from K such that infinity is an example of holistic property in this case.

2. Suppose X, Y, Z … are sets from R^n, where R^n is n-dimensional real space, and T = (R^n,τ) is the usual topology in R^n. Let us introduce the following predicates on the sets.

nSet(X) is “X is a set from R^n”

OpSet(X) is the assertion “X is an open set in T”.

PClSet(X) is the assertion “X is a proper close set in T”, where PClSet(X) is equivalent to the assertions “X is a close set in T” and “there no exists an open set Y in T such that Y contains X”.

Put by definition

$X \le Y \equiv X \subseteq Y \wedge nSet(X) \wedge nSet(Y)$,

$X \le^1 Y \equiv X \subseteq Y \wedge PClSet(X) \wedge PClSet(Y)$,

$X \le^2 Y \equiv X \subseteq Y \wedge OpSet(X) \wedge OpSet(Y)$.

Therefore, 1-level is the class of all proper close sets in topology T. 2-level is the class of all open sets in T. The reader will easily prove that axioms (AH1) and (AH2) hold in this case too. The following theorem can be proved.

Theorem of Opn-whole. $Pos^2(X) \equiv OpSet(X) \wedge \neg(X=\varnothing)$

Therefore, wholes are nonzero open sets in T such that nonzero openness (continuum) is an example of one more holistic property.

3. Suppose X, Y, Z … are sets from R^n, where R^n is n-dimensional real space, and M = (R^n,μ^n) is a metric space on R^n with, for example, Borelean n-dimensional measure μ^n. Let us introduce the following predicates on the sets.

nMSet(X) is the assertion “X is a μ^n-measured set from R^n”.

PnMSet(X) is the assertion “$\mu^n(X) > 0$ and X is a μ^n-measured set from R^n”.

NnMSet(X) is the assertion “$\mu^n(X) = 0$ and X is a μ^n-measured set from R^n”.

Put by definition

$X \le Y \equiv X \subseteq Y \wedge nMSet(X) \wedge nMSet(Y)$,

$X \le^1 Y \equiv X \subseteq Y \wedge NnMSet(X) \wedge NnMSet(Y)$,

$X \le^2 Y \equiv X \subseteq Y \wedge (PnMSet(X) \vee X=\varnothing) \wedge (PnMSet(Y) \vee Y=\varnothing)$.

Hence 1-level is the class of all μ^n-measured set from R^n with zero measure μ^n. 2-level is the class of all μ^n-measured sets from R^n with nonzero measure μ^n and null set. Axioms (AH1) and (AH2) hold in this case also. The following theorem can be proved.

Theorem of PnM-whole. $Pos^2(X) \equiv PnMSet(X)$

Therefore, wholes are nonzero μ^n-measured sets in R^n such that nonzero n-dimensional measureness is the third example of holistic property.

MLW can be extended by different additional axioms that can express more particular kinds of wholes. These extended versions could form a specter of different Logics of Wholes (LW). However MLW might be the intersection, the minimum general part, of all LW's. MLW expresses ideas of two-level order: holistic status of the second level in the respect to the first level, the idea of trans-level order, etc.

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